## Hölder spaces on Riemannian manifolds

Let $(M,g)$ be a Rie. manifold. For $k\in N$, we define the space $C^k(M)$ by

$$C^k(M)=\{u: M\to R:\ u \text{ k-times continuously differentiable function},\ \|u\|_{C^k}<\infty\}$$

where $\|\cdot\|_{C^k}$ is defined by

$$\|u\|_{C^k}=\sum_{j=0}^{k}\sup_{x\in M}|\nabla^j u(x)|$$

Remark: $C^k(M)$ is a Banach space.

In the regularity theory for elliptic and parabolic partial differential equations it is more convenient to work with Hölder spaces than with $C^k$-spaces, since these turn out to have better regularity properties.

Let $\alpha\in(0,1)$ and $T$ be a tensor field over $M$. Then we define a seminorm

$$[T]_\alpha=\sup_{\substack{x\neq y\in M\\ d_g(x,y)<\delta_g(x)}}\frac{|T(x)-T(y)|}{d_g(x,y)^\alpha}$$

Here $\delta_g(x)$ denotes the injectivity radius of $g$ at $x$. Moreover, $|T(x)-T(y)|$ is understood in the sense that we first take the parallel transport of $T(x)$ along the unique minimizing geodesic connecting $x$ and $y$, and then compute the norm at the point $y$. We define the $C^{k,\alpha}$-norm by

$$\|u\|_{C^{k,\alpha}}=\|u\|_{C^k}+[\nabla^k u]_\alpha$$

The number $\alpha$ is called the Hölder exponent.

We define the Hölder space $C^{k,\alpha}(M)$ by

$$C^{k,\alpha}(M):=\{u\in C^k(M):\ \|u\|_{C^{k,\alpha}}<\infty\}$$

Remark1: 由于平行移动是保距的，将T从y处拉到x处与T(x)作内积与将T从x处拉到y处与T(y)作内积是一样的.

Remark2: $C^{k,\alpha}(M)$ is a Banach space.

Adjoint

On $\mathbb{R}^1$ with the $L^2$ inner product, the adjoint of $D=\frac{d}{dx}$ is found simply by integrating by parts: for all $\varphi, \chi \in C_c^\infty(\mathbb{R}^1)$

$\langle \varphi, D\chi \rangle = \int_{\mathbb{R}^1} \varphi \overline{\chi'}\, dx = -\int_{\mathbb{R}^1} \varphi' \overline{\chi}\, dx = \langle -D\varphi, \chi \rangle$.

Thus, the adjoint of $\frac{d}{dx}$ is $-\frac{d}{dx}$.

More correctly, because $\frac{d}{dx}$ is an unbounded operator on $L^2$ and thus not defined on the whole Hilbert space, this is the formal adjoint.

Remark: Laplacian is self-adjoint on $H_0^{2,2}(M)$.

Let $E$ and $F$ are smooth Hermitian vector bundles over $M$, and let $L: C^\infty(E) \to C^\infty(F)$ be a $\mathbb{C}$-linear map. We say that $L$ is a differential operator if for any choice of local coordinates $(U; x^i)$ and local section frame $(U; e_i)$ of $E$, local section frame $(U; f_j)$ of $F$, we write $u = u^i e_i$, then $Lu = \sum_{|\alpha| \le k} a^{\alpha}{}_i{}^j(x)\, \partial^\alpha u^i f_j$

我们来验证 differential operator 的定义是良定的, 即不依赖于局部 section frame 的选取.

$\{\tilde{e}_j\}$ 另一个 local section frame of $E$ $\quad\{\tilde{f}_j\}$ 另一个 local section frame of $F$

$e_i = a_i{}^j \tilde{e}_j, \quad f_j = b_j{}^k \tilde{f}_k \qquad u = u^i e_i = u^i a_i{}^j \tilde{e}_j = \tilde{u}^j \tilde{e}_j$

$$\begin{aligned} Lu &= \sum_{|\alpha| \le k} a^{\alpha}{}_i{}^j(x)\, \partial^\alpha \left( (a^{-1})_l{}^i\, \tilde{u}^l \right) b_j^k(x) \tilde{f}_k \\ &= \sum_{|\alpha| \le k} a^{\alpha}{}_i{}^j(x) \sum_{\beta+\gamma=\alpha} \partial^\beta \left( (a^{-1})_l{}^i \right) \partial^\gamma \tilde{u}^l \cdot \frac{\alpha!}{\beta!\,\gamma!}\, b_j^k(x) \tilde{f}_k \\ &= \sum_{|\alpha| \le k} \sum_{\beta+\gamma=\alpha} \underline{a^{\alpha}{}_i{}^j(x)\, \partial^\beta \left( (a^{-1})_l{}^i \right) b_j^k(x) \frac{\alpha!}{\beta!\,\gamma!}}\; \partial^\gamma \tilde{u}^l\, \tilde{f}_k \end{aligned}$$

副主部: 最高阶不变

取 $\gamma$ 满足 $|\gamma| = k$, 则 $\beta$ 必为 0

$\Rightarrow \underbrace{a_i^{\gamma j}(x)\, b_j^k(x)}_{\text{可逆}}$

于是我们可以定义:

A differential operator is said to be of order $k$ if there are no derivatives of order $\ge k+1$ appearing in a local representation.

Assume that $P: C^\infty(E) \to C^\infty(F)$ is a linear differential operator, then one can use the $L^2$ inner product to define the formal adjoint, $P^*$, by the usual rule $\langle Pu, v\rangle_F = \langle u, P^* v\rangle_E$

for all smooth sections $u \in C^\infty(E)_c$ and $v \in C^\infty(F)_c$. Since the supports of $u$ and $v$ can be assumed to be in a coordinate patch, one can compute $P^*$ locally using integration by parts.

locally, $Pu = \sum_{|\alpha|\le k} \underbrace{a^\alpha(x)}_{\text{矩阵}} \underbrace{\partial^\alpha u}_{\text{列向量}}$.

$$\Rightarrow P^* v = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \frac{\partial^\alpha (a^\alpha(x)^* v \sqrt{G})}{\sqrt{G}}$$

where $\sqrt{G}\,dx$ volume form of $M$.

ATTENTION: 这是在E,F的两基下的形式. 一般情况下, $P^*$的局部表述中会出现向量丛E,F的度量系数 i.e. 若$\{e_i\}$为一般的局部标架 $g_{ij} := \langle e_i, e_j\rangle_E$.

Principal Symbol

For a linear constant coefficient differential operator

$$Pu = \sum_{|\alpha|\le k} a^\alpha \partial^\alpha u.$$

a standard approach to solving $Pu = f$ is to use Fourier analysis.

Then, say on $\mathbb{R}^n$, taking the Fourier transform gives

$$P(\xi)\hat{u} = \hat{f}$$

where $$P(\xi) = \sum_{|\alpha|\le k} i^{|\alpha|} a^\alpha \xi^\alpha$$

is an ordinary polynomial in $\xi$.

形式地, $u = \mathcal{F}^{-1}\left(\frac{1}{P(\xi)} \hat{f}\right)$.

For the variable coefficient case, one can obtain useful information by freezing the coefficients at one point and examining the corresponding constant coefficient case. This leads one to define the symbol of a linear differential operator.

$P: C^\infty(E) \to C^\infty(F)$, $x \in M$, $\xi \in T_x^* M$, $Pu = \sum_{|\alpha|\le k} a^\alpha(x) \partial^\alpha u$ locally

one can associate an algebraic object, the principle symbol $\sigma_\xi(P; x)$

$\sigma_\xi(P, x) = i^k \sum_{|\alpha| = k} a^\alpha(x) \xi^\alpha$. $\sigma_\xi(P, x)$ 为 $\dim F \times \dim E$ 阶矩阵

Remark: $\sigma_\xi(p,x)$ 的定义是借助于 $P$ 的局部表达来定义的. 这样定义方便计算 但是一个"nice"的定义不应依赖于局部表达. 为了看到这一点, 我们给出 symbol 的另个定义.

一方面 local section frames of $E$ and $F$ 另一方面 $\xi = \xi^i dx^i$

Let $E_x$ and $F_x$ be the fibers of $E$ and $F$ at $x \in M$, let $u \in C^\infty(E)$ with $u(x) = z$, and let $\varphi \in C^\infty(M)$ with $\varphi(x) = 0$ $d\varphi(x) = \xi$, then $\sigma_\xi(p;x): E_x \to F_x$ is the following endomorphism

$$\sigma_\xi(p;x)z = \frac{i^k}{k!} P(\varphi^k u)|_x$$

证明: 两个定义一致. 第二个定义不依赖于 $u, \varphi$ 的选择.

$$P(\varphi^k u) = \sum_{|\alpha| \le k} a^\alpha(x) \partial^\alpha(\varphi^k u) = \sum_{|\alpha|\le k} a^\alpha(x) \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial^\beta(\varphi^k)\partial^\gamma u$$

$$\Rightarrow P(\varphi^k u)|_x = \sum_{|\alpha|=k} a^\alpha(x)\, k!\, \xi^\alpha u(x) = \sum_{|\alpha|=k} a^\alpha(x)\, k!\, \xi^\alpha z$$

$$\Rightarrow \sigma_\xi(p;x) z = i^k \sum_{|\alpha|=k} a^\alpha(x)\xi^\alpha z \quad ✓$$

Remark: To illustrate the value of the principal symbol, shortly we will use an algebraic property of $\sigma_\xi(p)$ to define an elliptic differential operator. This algebraic property of ellipticity then will implies analytic conclusions.

prop (symbol 的代数性质).

(i) $\sigma_\xi(p+Q) = \sigma_\xi(p) + \sigma_\xi(Q)$ $\qquad$ $P, Q$ 有相同的阶

(ii) $\sigma_\xi(pQ) = \sigma_\xi(p)\sigma_\xi(Q)$ $\qquad$ $C^\infty(E) \xrightarrow{Q} C^\infty(F) \xrightarrow{P} C^\infty(G)$

(iii) $\sigma_\xi(p^*) = \sigma_\xi(p)^*$ $\longleftarrow$ symbol 的定义 $i^k$ 的作用是保证 (iii) 成立.

例 1. $(M,g)$ the exterior derivative $d$.

$$d: \Omega^p(M) \to \Omega^{p+1}(M)$$

由于 $d(\varphi\alpha) = d\varphi\wedge\alpha + \varphi\, d\alpha$ $\qquad$ $\varphi \in C^\infty(M)$. $\alpha \in \Omega^p(M)$

由 symbol 的第二个定义 知 $\sigma_\xi(d)\alpha = i\xi\wedge\alpha$.

means $\sigma_{d\varphi(x)}(d)\,\alpha|_x = i\, d\varphi(x)\wedge\alpha|_x \quad \forall x\in M$

2. the covariant derivative $D: C^\infty(E) \to \Omega^1(E)$ satisfies

$D(\varphi v) = d\varphi\otimes v + \varphi Dv$ $\qquad$ $\varphi\in C^\infty(M)$. $v \in C^\infty(E)$

$$\Rightarrow \sigma_\xi(D)v = i\xi\otimes v$$

means $\sigma_{d\varphi(x)}(D)\, v|_x = i\, d\varphi(x)\otimes v|_x \quad \forall x \in M$.